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Multimomentum Hamiltonian Formalism in Quantum Field Theory

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The familiar generating functionals in quantum field theory fail to be true measures and make sense only in framework of perturbation theory. In our approach, generating functionals are defined strictly as the Fourier transforms of Gaussian measures in nuclear spaces of multimomentum canonical variables when field momenta correspond to derivatives of fields with respect to all world coordinates, not only to time.

1. INTRODUCTION

Contemporary field models almost always have constraints. In order to describe them, one can apply the covariant multimomentum generalization of the Hamiltonian formalism in mechanics (Sardanashvily and Zakharov, 1992*a*, *b*, 1993). The multimomentum canonical variables are field functions ϕ^i and momenta p_i^λ associated with derivatives of ϕ^i with respect to all world coordinates x^μ, not only the time.

In classical field theory, if a Lagrangian density is degenerate, the system of the Euler–Lagrange equations becomes underdetermined and requires additional conditions. In gauge theory, these are gauge conditions which single out a representative from each gauge class. In the general case, the above-mentioned supplementary conditions remain elusive. In the framework of the multimomentum Hamiltonian formalism, one obtains them automatically because a part of the Hamilton equations play the role of gauge conditions. The key point consists in the fact that, given a degenerate Lagrangian density, one must consider a family of associated multimomentum Hamiltonian forms in order to exhaust solutions of the Euler–Lagrange equations (Sardanashvily and Zakharov, 1993). There

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exist comprehensive relations between Lagrangian and multimomentum Hamiltonian formalisms for degenerate quadratic and affine densities. Most field models are of these types. As a result, we get a general procedure of describing constraint systems in classical field theory.

The present work is devoted to the multimomentum quantum field theory. This theory, like the well-known current algebra models, has been hampered by the lack of satisfactory commutation relations between multimomentum canonical variables (Günter, 1987; Cariñena *et al.*, 1991). We base this work on the fact that the operation of chronological product of quantum bosonic fields is commutative and so Euclidean chronological forms can be represented by states on commutative tensor algebras. Therefore, restricting our consideration to generating functionals of Green functions, we can overcome the difficulties of establishing the multimomentum commutation relations.

Moreover, the multimomentum quantum field theory may incorporate the canonical and algebraic approaches to the quantization of fields. In physical models, the familiar expression

$$N^{-1}\exp\left[i\int L(\phi)\right]\prod_x [d\phi(x)]$$

of a generating functional fails to be a true measure since the Lebesgue measure in infinite-dimensional linear spaces is not defined in general.

In algebraic quantum field theory, generating functionals of chronological forms result from the Wick rotation of the Fourier transforms of Gaussian measures in the duals to nuclear spaces (Sardanashvily and Zakharov, 1991; Sardanashvily, 1991). The problem has consisted in constructing such measures. In the present work, we get these measures in terms of multimomentum canonical variables. They have the universal form due to the canonical splitting of multimomentum Hamiltonian forms. In particular, we reproduce the Euclidean propagators of scalar fields and gauge potentials. Note that the covariant multimomentum canonical quantization can be generalized to any field model with a degenerate quadratic Lagrangian density.

2. MULTIMOMENTUM HAMILTONIAN FORMALISM

We consider the multimomentum generalization of the familiar Hamiltonian formalism to fibered manifolds $\pi: E \to X$ over an n-dimensional base X, not only $X = \mathbb{R}$. If sections of E describe classical fields, one can apply this formalism to field theory. In this case, the Legendre manifold

$$\Pi = \overset{n}{\wedge} T^*X \underset{E}{\otimes} TX \underset{E}{\otimes} V^*E \tag{1}$$

plays the role of the finite-dimensional phase space of fields. By VE and V^*E are meant the vertical tangent and cotangent bundles over a fibered manifold E. Given an atlas of fibered coordinates (x^μ, y^i) of E, the Legendre manifold is provided with the linear adapted coordinates (x^μ, y^i, p_i^μ). In these coordinates, a multimomentum Hamiltonian form on Π and the corresponding Hamilton equations read

$$H = p_i^\lambda\, dy^i \wedge \omega_\lambda - \mathscr{H}\omega = p_i^\lambda\, dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i \omega - \tilde{\mathscr{H}}\omega \tag{2}$$

$$\partial_\lambda y^i = \partial_\lambda^i \mathscr{H}, \qquad \partial_\lambda p_i^\lambda = -\partial_i \mathscr{H} \tag{3}$$

$$\omega = dx^1 \wedge \cdots \wedge dx^n, \qquad \omega_\lambda = \partial_\lambda \rfloor \omega$$

where Γ is a connection on E and $\tilde{\mathscr{H}}\omega$ is a horizontal density on $\Pi \to X$.

The multimomentum Hamiltonian formalism is associated with the Lagrangian formalism in jet manifolds where the jet manifold J^1E of E plays the role of a finite-dimensional configuration space (Bauderon, 1982; Giachetta and Mangiarotti, 1989). The jet manifold J^1E comprises classes $j_x^1\phi$ of sections ϕ of E which are identified by the first two terms of their Taylor series at points x. It is provided with the adapted coordinates $(x^\lambda, y^i, y_\lambda^i)$, where

$$y_\lambda^i(j_x^1\phi) = \partial_\lambda \phi^i(x)$$

A first-order Lagrangian density $L = \mathscr{L}(x^\lambda, y^i, y_\lambda^i)\omega$ on J^1E defines the Legendre morphism $\hat{L}$ of J^1E to Π:

$$(x^\lambda, y^i, p_i^\lambda) \circ \hat{L} = (x^\lambda, y^i, \pi_i^\lambda), \qquad \pi_i^\lambda = \partial_i^\lambda \mathscr{L}$$

Conversely, a multimomentum Hamiltonian form H on Π defines the momentum morphism $\hat{H}$ of Π to J^1E:

$$(x^\lambda, y^i, y_\lambda^i) \circ \hat{H} = (x^\lambda, y^i, \partial_\lambda^i \mathscr{H})$$

We say that a multimomentum Hamiltonian form H is associated with a Lagrangian density L if

$$\hat{L} \circ \hat{H}|_Q = \mathrm{Id}\, Q, \qquad Q = \hat{L}(J^1E)$$

$$\mathscr{L}(x^\mu, y^i, \partial_\mu^i \mathscr{H}) = p_i^\lambda\, \partial_\lambda^i \mathscr{H} - \mathscr{H}$$

In general, different multimomentum Hamiltonian forms may be associated with the same Lagrangian density. Most field models meet the following relations between Lagrangian and multimomentum Hamiltonian formalisms.

(i) All multimomentum Hamiltonian forms H associated with a Lagrangian density L are equal to each other on the constraint space Q, that is, $H|_Q = H_L$. Moreover, for every section ϕ of E, we have

$$(\hat{L} \circ j^1\phi)^*(H_L) = (j^1\phi)^*(L) = L(\phi) \tag{4}$$

(ii) If a solution r of the Hamilton equations (3) corresponding to a multimomentum Hamiltonian form H associated with a Lagrangian density L belongs to the constraint space Q, then $\hat{H} \circ r$ is a solution of the Euler–Lagrange equations for L. Conversely, for each local solution s of the Euler–Lagrange equations defined by a Lagrangian density L, there exists an associated multimomentum Hamiltonian form such that $\hat{L} \circ s$ is a solution of the corresponding Hamilton equations.

The relation (4) gives us the reason to use sections r of the Legendre manifold $\Pi \to X$ as functional variables in quantum field theory. On the physical level, one can consider the naive generating functional

$$Z = N^{-1} \int \exp\left[i \int (r^*H + \alpha_i \phi^i \omega + \alpha^i_\mu p^\mu_i \omega) \right] \prod_x [d\phi^i(x)][dp^\mu_i(x)] \tag{5}$$

$$r^*H = (p^\mu_i(x)\, \partial_\mu \phi^i(x) - \mathscr{H})\omega$$

Note that the canonical splitting (2) of multimomentum Hamiltonian forms leads to standard terms $p^\mu_i\, \partial_\mu \phi^i$ in generating functionals in multimomentum canonical variables.

The generating functional (5) fails to be a true measure. The problem of representation of generating functional by measures can be settled in the framework of algebraic quantum field theory (Sardanashvily and Zakharov, 1991; Sardanashvily, 1991).

3. ALGEBRAIC QUANTUM FIELD THEORY

In accordance with the algebraic approach, a quantum field system can be characterized by a topological *-algebra A and by a continuous state f on A. To describe particles, one considers usually a tensor algebra A_Φ of a real, linear, locally convex topological space Φ endowed with the involution operation. We further assume that Φ is a nuclear space.

In the axiomatic quantum field theory of real scalar fields, the quantum field algebra is A_Φ with $\Phi = \mathbb{R}S_4$. By $\mathbb{R}S_m$ is meant the real subspace of the nuclear Schwartz space $S(\mathbb{R}^m)$ of complex functions $\phi(x)$ on $\mathbb{R}^m$ such that

$$\|\phi\|_{k,l} = \max_{|\alpha| \le l,\, x\in\mathbb{R}^m} \sup\, (1+|x|)^k \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^m)^{\alpha_m}} \phi(x), \qquad |\alpha| = \alpha_1 + \cdots + \alpha_m$$

is finite for any collection $(\alpha_1, \ldots, \alpha_m)$ and all $l, k \in \mathbb{Z}^{\ge 0}$. A state f on $A_{\mathbb{R}S_4}$

is represented by a family of temperature distributions $W_n \in S'(\mathbb{R}^{4n})$:

$$f(\phi_1 \cdots \phi_n) = \int W_n(x_1, \ldots, x_n)\phi_1(x_1) \cdots \phi_n(x_n)\, d^4x_1 \cdots d^4x_n$$

If f obeys the Wightman axioms, W_n are the familiar n-point Wightman functions.

To describe particles created at some moment and destructed at another moment, one uses the chronological forms f^c given by

$$W_n^c(x_1, \ldots, x_n) = \sum_{(i_1 \cdots i_n)} \theta(x_{i_1}^0 - x_{i_2}^0) \cdots \theta(x_{i_{n-1}}^0 - x_{i_n}^0) W_n(x_1, \ldots, x_n) \quad (6)$$

where $(i_1 \cdots i_n)$ is a rearrangement of numbers $1, \ldots, n$. The forms (6) fail to be distributions and do not define a state on $A_{\mathbb{R}S_4}$. At the same time, they issue from the Wick rotation of the Euclidean states on $A_{\mathbb{R}S_4}$ describing particles in the interaction zone.

Since chronological forms (6) are symmetric, Euclidean forms can be introduced as states on a commutative tensor algebra. Note that they differ from the Schwinger functions associated with the Wightman functions.

Let B_Φ be the commutative quotient of A_Φ. This algebra can be regarded as the enveloping algebra of the Lie algebra associated with the Lie commutative group G_Φ of translations in Φ. We therefore can construct a state on the algebra B_Φ as a vector form of its cyclic representation induced by a strong-continuous unitary cyclic representation of G_Φ. Such a representation is characterized by a positive-type continuous generating function Z on Φ, that is,

$$Z(\phi_i - \phi_j)\alpha^i \bar{\alpha}^j \geq 0, \qquad Z(0) = 1$$

for all collections of $\phi_1, \ldots, \phi_n$ and complex numbers $\alpha^1, \ldots, \alpha^n$. If the function $\alpha \to Z(\alpha\phi)$ on $\mathbb{R}$ is analytic at 0 for each $\phi \in \Phi$, the positive continuous form F on B_Φ is given by

$$F_n(\phi_1 \cdots \phi_n) = i^{-n} \frac{\partial}{\partial \alpha^1} \cdots \frac{\partial}{\partial \alpha^n} Z(\alpha^i \phi_i)|_{\alpha^i = 0}$$

In virtue of a well-known theorem (Gelfand and Vilenkin, 1964), any function Z of the above-mentioned type is the Fourier transform of a positive bounded measure μ in the dual Φ' to Φ:

$$Z(\phi) = \int_{\Phi'} \exp[i\langle w, \phi \rangle]\, d\mu(w) \quad (7)$$

where $\langle\ ,\ \rangle$ denotes the contraction between Φ' and Φ. The corresponding representation of G_Φ is given by operators

$$g(\phi): \quad u(w) \to \exp[i\langle w, \phi \rangle] u(w)$$

in the space of quadratically μ-integrable functions $u(w)$ on Φ', and we have

$$F_n(\phi_1 \cdots \phi_n) = \int \langle w, \phi_1 \rangle \cdots \langle w, \phi_n \rangle \, d\mu(w)$$

For instance, a generating function Z of a Gaussian state F on B_Φ reads

$$Z(\phi) = \exp\left[-\frac{1}{2} \Lambda(\phi, \phi) \right] \tag{8}$$

where the covariance form $\Lambda(\phi_1, \phi_2)$ is a positive-definite Hermitian bilinear form on Φ, continuous in ϕ_1 and ϕ_2. This generating function is the Fourier transform of a Gaussian measure in Φ'. The forms $F_{n>2}$ obey the Wick rules, where

$$F_1 = 0, \qquad F_2(\phi_1, \phi_2) = \Lambda(\phi_1, \phi_2)$$

In particular, if $\Phi = \mathbb{R}S_n$, the covariance form of a Gaussian state is uniquely defined by a distribution $W \in S'(\mathbb{R}^{2n})$:

$$\Lambda(\phi_1, \phi_2) = \int W(x_1, x_2)\phi_1(x_1)\phi_2(x_2) \, d^n x_1 \, d^n x_2$$

In field models, a generating function Z plays the role of a generating functional represented by the functional integral (7). If Z is the Gaussian generating function (8), its covariance form Λ defines Euclidean propagators. Propagators of fields in the Minkowski space are reconstructed by the Wick rotation of Λ (see appendix).

4. SCALAR FIELDS

Let E be a vector bundle over a world manifold X^4. Its sections describe scalar matter fields. In jet terms, their Lagrangian density reads

$$L_{(m)} = \frac{1}{2} a^E_{ij}[g^{\mu\nu}(y^i_\mu - \Gamma^i_\mu)(y^j_\nu - \Gamma^j_\nu) - m^2 y^i y^j]|g|^{1/2}\omega$$

$$\Gamma^i_\mu = \Gamma_\mu{}^i{}_j(x)y^j, \qquad g = \det g_{\mu\nu}$$

where a^E is a fiber metric in E, Γ is a linear connection on E, and g is a world metric on X^4. Because of the canonical vertical splitting $VE = E \times E$, the corresponding Legendre manifold (1) is

$$\overset{n}{\bigwedge} T^*X \underset{E}{\otimes} TX \underset{E}{\otimes} E^*$$

The Legendre morphism $\hat{L}_{(m)}$ and the unique multimomentum Hamiltonian form associated with the Lagrangian density $L_{(m)}$ are given by the expressions

$$p_i^\lambda \circ \hat{L}_{(m)} = g^{\lambda\mu} a_{ij}^E (y_\mu^j - \Gamma_\mu^j)$$

$$H_{(m)} = p_i^\lambda \, dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i(\mu) - \frac{1}{2}(a_E^{ij} g_{\mu\nu} p_i^\mu p_j^\mu |g|^{-1} + m^2 a_{ij}^E y^i y^j)|g|^{1/2}\omega \tag{9}$$

where a_E is the fiber metric in E^* dual to a^E.

For the sake of simplicity, we here examine scalar fields without symmetries. Let $\tilde{\phi}$ be real Euclidean scalar fields on the Euclidean space $X = \mathbb{R}^4$. The corresponding Legendre manifold

$$\tilde{\Pi} = \left(\bigwedge^4 T^*X \underset{X}{\otimes} TX\right) \underset{X}{\times} \mathbb{R}$$

is provided with the adapted coordinates $(z^\mu, \tilde{y}, \tilde{p}^\mu)$. Sections r of $\tilde{\Pi}$ are represented by functions $(\tilde{\phi}(z), \tilde{p}^\mu(z))$ on $\mathbb{R}^4$ which take their values in the vector space

$$V = \left(\bigwedge^4 \mathbb{R}_4 \otimes \mathbb{R}^4\right) \times \mathbb{R}, \qquad \mathbb{R}_4 = (\mathbb{R}^4)'$$

Their commutative tensor algebra is B_Φ, where $\Phi = V \otimes \mathbb{R}S_4$. The scalar form

$$\langle r|r\rangle_\Phi = \int [\delta_{\mu\nu}\tilde{p}^\mu(z)\tilde{p}^\nu(z) + \tilde{\phi}^2(z)]\, d^4z$$

brings Φ into the rigged Hilbert space. Let

$$H_{(m)} = \tilde{p}^\mu \, d\tilde{y} \wedge \omega_\mu - \frac{1}{2}(-\delta_{\mu\nu}\tilde{p}^\mu\tilde{p}^\nu + m^2\tilde{y}^2)\omega$$

be the multimomentum Hamiltonian form describing Euclidean scalar fields. The covariance form Λ of the associated generating function is defined by the relation

$$\int 2r^* H_{(m)} = \langle r|\gamma r\rangle_\Phi = -\Lambda(\gamma r, \gamma r), \qquad r \in \Phi \tag{10}$$

where γ is the first-order linear differential operator on Φ. We have

$$\Lambda(r, r) = \int \left[\Delta_F(z_1, z_2)\tilde{\phi}(z_1)\tilde{\phi}(z_2) + \tilde{p}^\mu(z_1)\frac{\partial \Delta_F}{\partial z_1^\mu}\tilde{\phi}(z_2) + \tilde{\phi}(z_1)\frac{\partial \Delta_F}{\partial z_2^\mu}\tilde{p}^\mu(z_2) \right.$$
$$\left. + \left(-\delta_{\mu\nu}\delta^{(4)}(z_1 - z_2) + \frac{\partial^2 \Delta_F}{\partial z_1^\mu \, \partial z_2^\nu}\right)\tilde{p}^\mu(z_1)\tilde{p}^\nu(z_2)\right] d^4z_1 \, d^4z_2 \tag{11}$$

$$\Delta_F(z_1, z_2) = \int \Delta_F(q) \exp[iq(z_1 - z_2)]\, d_4q, \qquad \Delta_F(q) = (m^2 + \delta^{\mu\nu}q_\mu q_\nu)^{-1}$$

where Δ_F is the Feynman propagator of Euclidean scalar fields.

Remark. The Schwartz space $S(\mathbb{R}^m)$ is the dense subset of $S'(\mathbb{R}^m)$. Being continuous on S_m, the scalar form $\langle\ |\ \rangle_{S_m}$ and the covariance forms Λ have no continuous prolongation to $S'(\mathbb{R}^m)$. In practice, one can consider prolongation of chronological forms to elements of $S'(\mathbb{R}^m)\backslash S(\mathbb{R}^m)$, which are the generalized eigenvectors of translation operators if the corresponding integrals converge.

5. GAUGE THEORY

Let $P \to X^4$ be a principal bundle with a structure Lie group G of internal symmetries. There is the 1:1 correspondence between principal connections on P and global sections A^C of the affine bundle $C = J^1P/G$ modeled on the vector bundle

$$\bar{C} = T^*X \otimes V^G P, \qquad V^G P = VP/G \tag{12}$$

The bundle C is provided with the fibered coordinates (x^μ, k^m_μ) such that its section A^C has the coordinate expression

$$(k^m_\mu \circ A^C)(x) = A^m_\mu(x)$$

where $A^m_\mu(x)$ are coefficients of a local connection 1-form. In gauge theory, sections A^C are treated as gauge potentials.

The configuration space of gauge potentials is the jet manifold J^1C. It is provided with the adapted coordinates $(x^\mu, k^m_\mu, k^m_{\mu\lambda})$. There exists the canonical splitting

$$\begin{gathered} J^1C = C_+ \underset{C}{\oplus} C_- = (J^2P/G) \underset{C}{\oplus} \left(\overset{2}{\wedge} T^*X \otimes V^G P\right) \\ (k^m_\mu, s^m_{\mu\lambda}, \mathcal{F}^m_{\lambda\mu}) = (k^m_\mu, k^m_{\mu\lambda} + k^m_{\lambda\mu} + c^m_{nl}k^n_\lambda k^l_\mu, k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl}k^n_\lambda k^l_\mu) \end{gathered} \tag{13}$$

where c^m_{nl} are the structure constants of the Lie algebra $\mathfrak{g}$ of the group G. In the coordinates (13), the conventional Yang–Mills Lagrangian density $L_{(A)}$ of gauge potentials is given by the expression

$$L_{(A)} = \frac{1}{4} a^G_{mn} g^{\lambda\mu} g^{\beta\nu} \mathcal{F}^m_{\lambda\beta} \mathcal{F}^n_{\mu\nu} |g|^{1/2} \omega \tag{14}$$

where a^G is a G-invariant metric in the Lie algebra $\mathfrak{g}$ and g is a world metric on X^4.

For gauge potentials, we have the Legendre manifold

$$\Pi = \overset{4}{\wedge} T^*X \otimes TX \underset{C}{\otimes} [C \times \bar{C}]^*$$

provided with the canonical coordinates $(x^\mu, k^m_\mu, p^{\mu\lambda}_m)$. This is a phase space of gauge potentials. It also has the canonical splitting

$$p_m^{\mu\lambda} = p_m^{(\mu\lambda)} + p_m^{[\mu\lambda]} = \frac{1}{2}(p_m^{\mu\lambda} + p_m^{\lambda\mu}) + \frac{1}{2}(p_m^{\mu\lambda} - p_m^{\lambda\mu})$$

The fibered manifold $\Pi \to X$ is the affine bundle modeled on the vector bundle

$$\overset{4}{\wedge} T^*X \underset{\bar{C}}{\otimes} V^*\bar{C} \to X \tag{15}$$

The Legendre morphism corresponding to the Lagrangian density (14) is

$$p_m^{(\mu\lambda)} \circ \hat{L}_{(A)} = 0 \tag{16a}$$

$$p_m^{[\mu\lambda]} \circ \hat{L}_{(A)} = a_{mn}^G g^{\lambda\alpha} g^{\mu\beta} \mathscr{F}_{\alpha\beta}^n |g|^{1/2} \tag{16b}$$

The multimomentum Hamiltonian forms associated with the Lagrangian density (14) read

$$H_B = p_m^{\mu\lambda}\, dk_\mu^m \wedge \omega_\lambda - p_m^{\mu\lambda} \bar{\Gamma}_{\mu\lambda}^m \omega - \tilde{\mathscr{H}}\omega$$

$$\tilde{\mathscr{H}} = \frac{1}{4} a_G^{mn} g_{\mu\nu} g_{\lambda\beta} p_m^{[\mu\lambda]} p_n^{[\nu\beta]} |g|^{-1/2} \tag{17}$$

$$\bar{\Gamma}_{\mu\lambda}^m = \frac{1}{2}[c_{nl}^m k_\lambda^n k_\mu^l + \partial_\mu B_\lambda^m + \partial_\lambda B_\mu^m - c_{nl}^m (k_\mu^n B_\lambda^l + k_\lambda^n B_\mu^l)] - \Gamma_{\mu\lambda}^\beta (B_\beta^m - k_\beta^m)$$

where B is some section of C, $\bar{\Gamma}$ is a connection on C, and $\Gamma_{\mu\lambda}^\beta$ are Christoffel symbols of world metric g. We have

$$H_B|_Q = p_m^{[\mu\lambda]}\, dk_\mu^m \wedge \omega_\lambda - \frac{1}{2} p_m^{[\mu\lambda]} c_{nl}^m k_\lambda^n k_\mu^l \omega - \tilde{\mathscr{H}}\omega$$

The Hamilton equations corresponding to the multimomentum Hamiltonian form (17) read

$$\partial_\lambda p_m^{\mu\lambda} = -c_{lm}^n k_\nu^l p_n^{[\mu\nu]} + c_{ml}^n B_\nu^l p_n^{(\mu\nu)} - \Gamma_{\lambda\nu}^\mu p_m^{(\lambda\nu)} \tag{18}$$

$$\partial_\lambda k_\mu^m + \partial_\mu k_\lambda^m = 2\bar{\Gamma}_{(\mu\lambda)}^m \tag{19}$$

plus equation (16b). On the constraint space (16a), equations (16b) and (18) are the familiar Yang–Mills equations. Equation (19) plays the role of gauge condition.

In algebraic quantum field theory, only fields forming a linear space are quantized. We therefore fix a background gauge potential B and consider deviation fields $\Omega = A^{\mathrm{C}} - B$ which are sections of the vector bundle (12). The corresponding Legendre bundle (15) is endowed with the adapted coordinates

$$(x^\mu, \bar{k}^m_\mu, p^{\mu\lambda}_m) = (x^\mu, k^m_\mu - B^m_\mu(x), p^{\mu\lambda}_m) \tag{20}$$

Sections r of the Legendre bundle (15) over the Euclidean space $\mathbb{R}^4$ are represented by functions $(\Omega^m_\mu(z), p^{\mu\lambda}_m(z))$ taking their values in the vector space

$$F = \left(\bigwedge^4 \mathbb{R}_4 \otimes \mathbb{R}^8 \otimes \mathfrak{g}^*\right) \times (\mathbb{R}_4 \otimes \mathfrak{g})$$

The commutative tensor algebra of r is B_Φ, where $\Phi = F \otimes \mathbb{R}S_4$. The nuclear space Φ is provided with the corresponding scalar form $\langle\,|\,\rangle_\Phi$, which brings Φ into the rigged Hilbert space.

To define a Gaussian state on this algebra, let us consider the multimomentum Hamiltonian form (17). In the coordinates (20), it reads

$$\bar{H}_B = p^{\mu\lambda}_m\, d\bar{k}^m_\mu \wedge \omega_\lambda - p^{\mu\lambda}_m \bar{\Gamma}^m_{\mu\lambda}\omega - \tilde{\mathscr{H}}_B\omega$$

$$\tilde{\mathscr{H}}_B = \frac{1}{4} a^{mn}_G g_{\mu\nu} g_{\lambda\beta} p^{[\mu\lambda]}_m p^{[\nu\beta]}_n |g|^{-1/2} + \frac{1}{2} p^{[\mu\lambda]}_m (\mathscr{F}_B{}^m_{\mu\lambda} + c^m_{nl}\bar{k}^n_\lambda \bar{k}^l_\mu) \tag{21}$$

$$\bar{\Gamma}_B{}^m_{\mu\lambda} = c^m_{nl} B^n_\lambda \bar{k}^l_\mu + \Gamma^\beta_{\mu\lambda}\bar{k}^m_\beta$$

where $\mathscr{F}_B$ is the strength for the background gauge potential B, and Γ_B is a connection on $\bar{C}$ associated with the principal connection B. One can use the multimomentum Hamiltonian form (21) in order to quantize the deviation fields Ω on $X = \mathbb{R}^4$.

For the sake of simplicity, let us assume that the structure group G is compact and simple ($a^{mn}_G = -2\delta^{mn}$). We have

$$r^*H_B = r^*H_1 + r^*H_2 = \left[\frac{1}{2}\delta_{\mu\nu}\delta_{\lambda\beta}\delta^{mn} p^{[\mu\lambda]}_m p^{[\nu\beta]}_n + p^{\mu\lambda}_m \nabla_\lambda \Omega^m_\mu\right] d^4z$$
$$-\frac{1}{2} p^{[\mu\lambda]}_m (\mathscr{F}_B{}^m_{\mu\lambda} + c^m_{nl}\bar{k}^n_\lambda \bar{k}^l_\mu)\, d^4z \tag{22}$$

where ∇ denotes the covariant derivative corresponding to the principal connection B.

To construct the associated Gaussian state, we use the quadratic part r^*H_1 of the form (22). The term r^*H_2 describes interaction considered by the perturbation theory.

The scalar form $\int r^*H_1$ on Φ, however, is degenerate. There are two ways for this difficulty to be overcome.

(i) In accordance with the conventional quantization scheme, we can restrict ourselves to sections r taking values in the constraint space (16a).

Since the form

$$r^*\bar{H}_B = \frac{1}{2}[\delta_{\mu\nu}\delta_{\lambda\beta}\delta^{mn}p_m^{[\mu\lambda]}p_n^{[\nu\beta]} + p_m^{[\mu\lambda]}\mathscr{F}^m_{\mu\lambda}]\, d^4z$$

$$= r^*H_0 - \frac{1}{2}p_m^{[\mu\lambda]}c^m_{nl}(\Omega^n_\lambda + B^n_\lambda)(\Omega^l_\mu + B^l_\mu)\, d^4z \qquad (23)$$

is degenerate, we must then consider the gauge orbit space Ξ_G, which is the quotient space Ξ of connections A^C by the group of gauge transformations. There exists a neighborhood N centered at the image of B in Ξ_G such that there is a local section $s_B: N \to \Xi$ whose values are elements $A^C \in \Xi$ satisfying the gauge condition

$$\delta^{\mu\nu}\nabla_\mu(A^C_\nu - B_\nu) = 0$$

(Mitter and Viallet, 1981). Hence, N is locally isomorphic to a Hilbert space. Neglecting here the Gribov ambiguity problem, let us assume that there exists a connection B such that s_B is a global section. If the bilinear part r^*H_0 of the form (23) induces a nuclear scalar form on Ξ'_G, one can construct the associated Gaussian measure μ in Ξ_G. If s_B is a μ-measurable morphism, there exists a measure μ_{GF} in Ξ which is the image of μ with respect to s_B. If this measure exists, it is concentrated in $s_B(\Xi_G) \subset \Xi$. We call it the gauge-fixing measure. In contrast with the naive expressions used in the gauge models, it is not the measure whose base is a Gaussian measure and density is the Faddeev–Popov determinant. Determinant densities are attributes of Lebesgue measures, which fail to be defined in the general case.

(ii) The first procedure fails for the general case of degenerate field systems, without gauge invariance. At the same time, one can insert additional terms quadratic in $p_m^{(\mu\nu)}$ into the multimomentum Hamiltonian form (21) which bring $\int r^*H_1$ into a nondegenerate scalar form. In the general case, we have

$$\bar{H}'_B = \bar{H}_B - h\omega, \qquad h = \frac{1}{2}[a_1\delta_{\mu\nu}\delta_{\lambda\beta}\delta^{mn}p_m^{(\mu\lambda)}p_n^{(\nu\beta)} + a_2(p_n^{\mu\mu})^2] \qquad (24)$$

where $a_1 \neq 0$ and a_2 are some constants. The Lagrangian $L'_{(A)}$ associated with the multimomentum Hamiltonian form (24) includes additional terms quadratic in $\bar{k}^m_{(\mu\nu)} + c^m_{nl}k^n_{(\mu}B^l_{\nu)}$. If the quadratic form $\int r^*H'_1$ is nondegenerate, one can use a relation similar to the relation (10) in order to construct the covariance form Λ_B of the associated Gaussian generating function Z_B.

For instance, if $a_1 = 1$ and $a_2 = 0$, we have

$$r^*H'_1 = \left[\frac{1}{2}\delta_{\mu\nu}\delta_{\lambda\beta}\delta^{mn}p_m^{\mu\lambda}p_n^{\nu\beta} + p_m^{\mu\nu}\nabla_\lambda\Omega^m_\mu\right] d^4z$$

The associated covariance form Λ_B exists. In particular, it defines the propagator of Euclidean deviation fields Ω which coincides with the Green's operator of the covariant Laplacian $\delta^{\mu\nu}\nabla_\mu\nabla_\nu$. This Green's operator exists (Mitter and Viallet, 1981). In the case of $B = 0$, the propagator of deviation fields Ω coincides with the familiar propagator of gauge potential which corresponds to the Feynman gauge ($\alpha = 1$), but there are no ghost fields. In comparison with the measure μ_{GF}, the Gaussian measure μ_B defined by the generating function Z_B is not concentrated in a gauge-fixing subset.

Note that, after gauge transformations $B \to B'$, measures μ_B and $\mu_{B'}$ are not equivalent in general. This means that a gauge phase of a background gauge potential may be valid, otherwise electromagnetic potentials. In the case of an Abelian structure group G, the multimomentum Hamiltonian form (23), the associated covariance form Λ_B, and the Gaussian measure μ_B are independent of a background potential B.

APPENDIX

We use the Fourier–Laplace (FL) transforms in order to construct strictly the Wick rotation.

Remark. By $\mathbb{R}^n_+$ and $\bar{\mathbb{R}}^n_+$ we denote the subset of $\mathbb{R}^n$ with the Cartesian coordinates $x^\mu > 0$ and its closure, respectively. Elements of $S(\mathbb{R}^n_+)$ are $\phi \in S(\mathbb{R}^n)$ such that $\phi = 0$ on $\mathbb{R}^n \backslash \mathbb{R}^n_+$. Elements of $S(\bar{\mathbb{R}}^n_+)$ correspond to distributions $W \in S'(\mathbb{R}^n)$ with supp $W \subset \bar{\mathbb{R}}^n_+$.

Given $W \in S'(\mathbb{R}^n)$, let Q_W be the set of $q \in \mathbb{R}_n$ such that $\exp(-qx)W \in S'(\mathbb{R}^n)$. The FL transform of W is defined to be the Fourier transform

$$W^{FL}(k + iq) = [\exp(-qx)W]^F = \int \exp[i(k + iq)x]\, W\, d^n x \in S'(\mathbb{R}_n)$$

It is a holomorphic function on the tabular set $\mathbb{R}_n + i\mathring{Q}_W \subset \mathbb{C}_n$ over the interiority of Q_W. Moreover, it defines the family of distributions $W^{FL}_q(k) \in S'(\mathbb{R}_n)$ which is continuous in the parameter q. In particular, if $W \in S'(\bar{\mathbb{R}}^n_+)$, then $\bar{\mathbb{R}}_{+n} \subset Q_W$ and W^{FL} is a holomorphic function on the tubular set $\mathbb{R}_n + i\mathbb{R}_{+n}$, so that $W^{FL}(k + i0) = W^F(k)$, i.e.,

$$\lim_{|q| \to 0, q \in \mathbb{R}_{+n}} \langle W^{FL}_q, \phi \rangle = \langle W^F, \phi \rangle, \qquad \phi(k) \in S(\mathbb{R}_n)$$

Let $W^{FL}(k + iq)$ be the FL transform of some distribution $W \in S'(\bar{\mathbb{R}}^n_+)$. Then, the relation

$$\int_{\mathbb{R}_{+n}} W^{FL}(iq)\phi(q)\, d_n q = \int_{\mathbb{R}^n_+} W(x)\hat{\phi}(x)\, d^n x, \qquad \phi \in S(\mathbb{R}_{+n})$$
$$\hat{\phi}(x) = \int_{\mathbb{R}_{+n}} \exp(-qx)\, \phi(q)\, d_n q, \qquad x \in \bar{\mathbb{R}}^n_+, \quad \hat{\phi} \in S(\bar{\mathbb{R}}^n_+) \tag{A1}$$

defines the continuous linear functional $W^L(q) = W^{FL}(iq)$ on $S(\mathbb{R}_{+n})$. It is called the Laplace transform. The image of $S(\mathbb{R}_{+n})$ under the continuous morphism $\phi \to \hat{\phi}$ is dense in $S(\bar{\mathbb{R}}^n_+)$, and the norms $\|\phi\|'_{k,l} = \|\hat{\phi}\|_{k,l}$ induce the weakening topology in $S(\mathbb{R}_{+n})$. The functional $W^L(q)$, (A1), is continuous with respect to this topology. There is a 1:1 correspondence between Laplace transforms of elements of $S'(\bar{\mathbb{R}}^n_+)$ and elements of $S'(\mathbb{R}_{+n})$ continuous with respect to the weakening topology in $S(\mathbb{R}_{+n})$. We use this correspondence in order to construct the Wick rotation.

If the Minkowski space is identified with the real subspace $\mathbb{R}^4$ of $\mathbb{C}^4$, its Euclidean partner is the subspace $(iz^0, x^{1,2,3})$ of $\mathbb{C}^4$. These spaces have the same spatial coordinate subspace $(x^{1,2,3})$. For the sake of simplicity, we henceforth do not write the spatial coordinate dependence. We consider the complex plane $\mathbb{C}^1 = X \oplus iZ$ of time x and Euclidean time z and the complex plane $\mathbb{C}_1 = K \oplus iQ$ of the associated momentum coordinates k and q.

Let $W(q) \in S'(Q)$ be a distribution such that

$$W = W_+ + W_-, \qquad W_+ \in S'(\bar{Q}_+), \qquad W_- \in S'(\bar{Q}_-) \tag{A2}$$

For every $\phi_+ \in S(X_+)$, we have

$$\begin{aligned}
\int_{\bar{Q}_+} W(q)\hat{\phi}_+(q)\,dq &= \int_{\bar{Q}_+} dq \int_{X_+} dx\,[W(q)\exp(-qx)\phi_+(x)] \\
&= \int_{\bar{Q}_+} dq \int_K dk \int_{X_+} dx\,[W(q)\phi^F_+(k)\exp(-ikx - qx)] \\
&\quad - i\int_{\bar{Q}_+} dq \int_K dk \left[W(q)\frac{\phi^F_+(k)}{k - iq}\right] \\
&= \int_{\bar{Q}_+} W(q)\phi^L_+(iq)\,dq
\end{aligned} \tag{A3}$$

due to the fact that the FL transform $\phi^{FL}_+(k + iq)$ of $\phi_+ \in S(X_+) \subset S'(X_+)$ exists and it is holomorphic on the tubular set $K + iQ_+$, $Q_+ \subset Q_{\phi_+}$, so that $\phi^{FL}_+(k + i0) = \phi^F_+(k)$. The function $\hat{\phi}_+(q) = \phi^{FL}_+(-q)$ can be regarded as the Wick rotation of $\phi_+(x)$. The relations (A3) take the form

$$\begin{aligned}
\int_{\bar{Q}_+} W(q)\hat{\phi}_+(q)\,dq &= \int_{X_+} \hat{W}_+(x)\phi_+(x)\,dx \\
\hat{W}_+(x) &= \int_{\bar{Q}_+} \exp(-qx)W(q)\,dq, \qquad x \in X_+
\end{aligned} \tag{A4}$$

where $\hat{W}_+(x) \in S'(X_+)$ is continuous with respect to the weakening topology in $S(X_+)$.

For every $\phi_- \in S(X_-)$, we have the similar relations

$$\int_{\bar{Q}_-} W(q)\hat{\phi}_-(q)\,dq = \int_{X_-} \hat{W}_-(x)\phi_-(x)\,dx$$
$$\hat{W}_-(x) = \int_{\bar{Q}_-} \exp(-qx)W(q)\,dq, \qquad x \in X_- \tag{A5}$$

The combination of (A4) and (A5) results in the relation

$$\int_Q W(q)\hat{\phi}(q)\,dq = \int_X \hat{W}(x)\phi(x)\,dx$$
$$\phi = \phi_+ + \phi_-, \qquad \hat{\phi} = \hat{\phi}_+ + \hat{\phi}_- \tag{A6}$$

where $\hat{W}(x)$ is a linear functional on functions $\phi \in S(X)$ such that ϕ and all its derivatives are equal to zero at $x = 0$. This functional can be regarded as a functional on $S(X)$, but it needs additional definition at $x = 0$. This is the well-known feature of chronological forms in quantum field theory. We can treat $\hat{W}$ as the Wick rotation of W.

For instance, let the covariance form Λ of a Gaussian state on the commutative algebra of Euclidean scalar fields $\tilde{\phi}$ be given by a distribution $\tilde{W}(z_1 - z_2)$. We have

$$\begin{aligned}
&\int \tilde{W}(z_1 - z_2)\tilde{\phi}_1(z_1)\tilde{\phi}_2(z_2)\,dz_1\,dz_2 \\
&\quad = \int \tilde{w}(z)\tilde{\phi}_1(z_1)\tilde{\phi}_2(z_1 - z)\,dz_1\,dz \\
&\quad = \int \tilde{w}(z)\tilde{f}(z)\,dz \\
&\quad = \int \tilde{w}^F(q)\tilde{f}^F(q)\,dq
\end{aligned}$$
$$z = z_1 - z_2, \qquad \tilde{f} = \int \tilde{\phi}_1(z_1)\tilde{\phi}_2(z_1 - z)\,dz_1$$

Let $\tilde{w}^F(q)$ satisfy the condition (26). Its Wick rotation (30) defines the functional

$$\begin{aligned}
\hat{w}(x) = \theta(x) &\int_{\bar{Q}_+} \tilde{w}^F(q)\exp(-qx)\,dq \\
&+ \theta(-x)\int_{\bar{Q}_-} \tilde{w}^F(q)\exp(-qx)\,dq
\end{aligned}$$
$$\int \hat{w}(x)f(x)\,dx = \int \hat{W}(x_1 - x_2)\phi(x_1)\phi(x_2)\,dx_1\,dx_2$$

on scalar fields ϕ on the Minkowski space. For instance, if $\tilde{w}^F(q)$ is the Feynman propagator (11), $i\hat{W}$ is the familiar causal Green function D^c.

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